

On Attractor Flow and Small Black Holes

Sanjay Siwach^{a*} and Bhupendra Nath Tiwari^{b†}

^a *Department of Physics,*

Banaras Hindu University,

Varanasi- 221 005, India.

and

^b*INFN-Laboratori Nazionali di Frascati*

Via E. Fermi, 40 - I-00044 Frascati

Rome, Italy.

We study the attractor flow and near horizon geometry of two-charge small black holes in heterotic string theory. The Hessian of Sen's entropy function with respect to the moduli fields has standard attractor properties and shows the interesting factorization at the attractor fixed points. We notice that the stability conditions are preserved under arbitrary α' -corrections to the black hole solutions.

*Electronic address: sksiwach@hotmail.com

†Electronic address: bntiwari.iitk@gmail.com

Black holes in string theory are the solutions of low energy supergravity actions. They obey the laws of black hole thermodynamics just like the black holes solutions in general theory of relativity. In particular, the macroscopic entropy is proportional to horizon area. It has been possible to count microstates for certain kind of black holes in string theory and in the large charge limit, the statistical entropy [1, 2] coincides with Bekenstein-Hawking entropy given in terms of the horizon area. The subleading higher derivative corrections are important particularly for small black holes for which the leading Bekenstein-Hawking entropy vanishes [3, 4].

Attractor mechanism [5–8] plays an important role in understanding the physics of string theory black holes. The scalar fields starting from their generic boundary values at radial infinity coupled to the black hole evolve towards their fixed values at the horizon. The fixed values are determined by the minima of black hole effective potential, which in general depends on the scalar fields and a set of invariant electric-magnetic charges. The attractor configurations, away from the critical points of the black hole potential also involve moduli fields at the asymptotic infinity [9]. The sensitivity of attractor flow to the higher derivative α' -corrections leads to the stringy insights into the moduli behavior away from attractor fixed points [10, 11]. This involves the effect of all possible field fluctuations against the macroscopic attractor configuration.

Entropy function formalism [12–19] seems to capture all higher derivative corrections to black hole entropy. The study of generalized attractor properties and associated nature of moduli in this formalism offers an unified interrelation of (i) black hole attractors, (ii) stringy α' -corrections and (iii) behavior of moduli flow equations. In general, this picture indicates non-trivial statistical configurations of the charges and moduli fields. In the present article, we shall focus our attention on the small black holes, and explicate the nature of attractor flow as moduli correlations.

We investigate the higher derivative α' -corrections for small black holes from the perspective of Sen’s entropy function. The attractor equations follow directly from the extremization of Sen’s entropy function [12–19] for a given set of higher derivative α' -corrections. The flow equations arising from the Hessian of Sen’s entropy function of the small black holes have been analyzed with respect to attractor moduli. The definition of the Hessian provides information about the covariant attractor flow on the moduli space.

In present paper, we shall examine these notions for two charge small black holes in het-

erotic string theory with arbitrary stringy corrections. The attractor properties of moduli are modified in general, but the attractor fixed point macroscopic configurations remain the same, in the present case. The higher order α' -corrections do not enter in the Hessian function $F_{ij}(u_S, u_T)$ of corrected entropy functions at any order for small black holes. We notice that Sen's entropy function and attractor flow at fixed points to be in generic agreement with the nature of the Hessian of BPS-mass factorization [20].

The extremal black holes in four spacetime dimensions [20–27] feature a near horizon geometry of the form $AdS_2 \times S^2$. Thus, the area of the horizon can be thus expressed in terms of (i) radius of S^2 , (ii) electric and magnetic charges, which in turn are determined as the minimum of the black hole effective potential V_{BH} , by the attractor mechanism [20–27]. It is interesting to analyze the attractor behavior of $V_{BH}(p, q, \phi^a)$, defined as the function of moduli fields and charges of the theory. To include the effect of asymptotic moduli fields [20], a generalization has been proposed as the negative Hessian function of the effective potential. The positivity of the Hessian provides stability condition leading to the critical behavior [20–27], at the extremum of V_{BH} .

We shall analyze black hole macroscopic configuration away from the attractor fixed point(s) by defining the flow equations as the negative Hessian of Sen entropy function;

$$g_{ij}^{(F)} = - \nabla_i \nabla_j F(p, q, u_i, v_i) \quad (1)$$

We shall be interested in attractor flow equations of the small black holes in heterotic string theory including all order corrections in α' . Considering the near horizon geometry of the form of $AdS_2 \times S^{D-2}$, the typical field configuration takes the following form;

$$\begin{aligned} ds^2 &= v_1(-r^2 dt^2 + \frac{dr^2}{r^2}) + v_2 d\Omega_{D-2}^2 \\ S &= u_S, \quad T = u_T \\ F_{rt}^{(i)} &= e_i; \quad i = 1, 2 \end{aligned} \quad (2)$$

Let us briefly summarize the Sen's entropy function formalism. For a generically covariant Lagrangian density \mathcal{L} , we define the horizon function;

$$f(\vec{u}, \vec{v}, \vec{e}) = \int_{S^{D-2}} \sqrt{-g} \mathcal{L}, \quad (3)$$

such that, the equations of motion in near horizon geometry Eqn.(2) read as;

$$\frac{\partial f}{\partial u_i} = 0, \quad \frac{\partial f}{\partial v_j} = 0; \quad (4)$$

The electric charges of the theory are defined as

$$q_i := \frac{\partial f}{\partial e_i} \quad (5)$$

Then, Sen's entropy function [12–17, 29? , 30] takes the form;

$$F(u_i, v_j, q_k) = 2\pi \left(\sum_{l=1,2} e_l q_l - f(u_i, v_j, e_l) \right). \quad (6)$$

Before proceeding further, it is worth to mention that there is nontrivial and unique choice of the small black hole parameters, independent of D , for which black hole entropy reduces to the statistical entropy $S = 4\pi\sqrt{nw}$, where n is KK momentum and w is winding charge. At finite order α' -corrections, there exists exact matching of the macroscopic attractor entropy and microscopic statistical entropy of the small black holes [30]. The analysis follows from consideration of 1/2-BPS states of heterotic string configuration compactified on $T^{9-D} \times S^1$ [31, 32].

For $AdS_2 \times S^{D-2}$ near horizon geometry, Prester [33] has shown that the horizon function reduces to

$$\begin{aligned} f(u_i, v_j, e_k) = & bu_S v_1 v_2^{(D-2)/2} \left\{ \frac{2u_T^2 e_1^2}{v_1^2} + \frac{2e_2^2}{u_T^2 v_1^2} \right. \\ & + \sum_{m=1}^{[D/2]} (\alpha'^{m-1} \lambda_m \frac{(D-2)!}{(D-2m)!} v_2^{-m} \\ & \left. [(D-2m)(D-2m-2) - 2m \frac{v_2}{v_1}] \right\}, \end{aligned} \quad (7)$$

where $\lambda_1 = 1$ and $b = \frac{\Omega_{D-2}}{16\pi G_N}$.

Consider the simplest case with $\{e_i\} \leftrightarrow \{q_i\}$. In $D = 4$ and $D = 5$ spacetime dimensions, this implies that the Sen entropy function remains unchanged for the small black holes with the following horizon function [33]

$$\begin{aligned} f(u_S, u_T, v_1, v_2, e_1, e_2) = & bu_S v_1 v_2^{(D-2)/2} \left[\frac{2u_T^2 e_1^2}{v_1^2} + \frac{2e_2^2}{u_T^2 v_1^2} \right. \\ & \left. - \frac{2}{v_1} \frac{(D-2)(D-3)}{v_2} \left(1 - \frac{4\alpha' \lambda_2}{v_1} \right) \right] \end{aligned} \quad (8)$$

Using Eqn.(5), we find that the associated charges of the theory are

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial e_1} = 4bu_S v_1^{-1} v_2^{(D-2)/2} u_T^2 e_1, \\ q_2 &= \frac{\partial f}{\partial e_2} = 4bu_S v_1^{-1} v_2^{(D-2)/2} u_T^{-2} e_2 \end{aligned} \quad (9)$$

For the small black hole in the above spacetime dimensions, it follows from Eqn.(6) that Sen entropy function has the following form;

$$F(u_S, u_T, v_1, v_2, q_1, q_2) = 2\pi \left[\frac{v_1}{8bu_S v_2^{(D-2)/2} u_T^2} q_1^2 + \frac{v_1 u_T^2}{8bu_S v_2^{(D-2)/2}} q_2^2 + 2bu_S v_2^{(D-2)/2} \right. \\ \left. - bu_S v_1 v_2^{(D-2)/2} \frac{(D-2)(D-3)}{v_2} \left(1 - \frac{4\alpha' \lambda_2}{v_1}\right) \right] \quad (10)$$

The corresponding Hessian derivatives of the above entropy function with respect to the moduli fields are;

$$g_{SS} = 2\pi \left(\frac{1}{4} \frac{v_1 q_1^2}{bS^3 v_2^{(D/2-1)} T^2} + \frac{1}{4} \frac{v_1 T^2 q_2^2}{bS^3 v_2^{(D/2-1)}} \right), \\ g_{ST} = 2\pi \left(\frac{1}{4} \frac{v_1 q_1^2}{bS^2 v_2^{(D/2-1)} T^3} - \frac{1}{4} \frac{v_1 T q_2^2}{bS^2 v_2^{(D/2-1)}} \right), \\ g_{TT} = 2\pi \left(\frac{3}{4} \frac{v_1 q_1^2}{bS v_2^{(D/2-1)} T^4} + \frac{1}{4} \frac{v_1 q_2^2}{bS v_2^{(D/2-1)}} \right), \quad (11)$$

where S and T denote the moduli fields u_S and u_T . In sequel, we also denote these flow components as g_{11} , g_{12} , and g_{22} . In this case, it is further known [33] that the attractor equations lead to the following horizon values of the spacetime parameters and moduli fields;

$$v_1 = 4\alpha' \lambda_2, \\ v_2 = 4(D-2)(D-3)\alpha' \lambda_2 m, \\ u_T = \sqrt{\frac{n}{w}}, \\ u_S = \frac{4\pi\alpha' G_N}{\Omega_{D-2}} \frac{v_1}{v_2^{(D-2)/2}} \sqrt{\frac{2nw}{\lambda_2}} \quad (12)$$

The corresponding value of electric fields are given as

$$e_1 = \sqrt{2\alpha' \lambda_2 \frac{n}{w}} \\ e_2 = \sqrt{2\alpha' \lambda_2 \frac{w}{n}} \quad (13)$$

Using the attractor values, viz., Eqn.(12), it has been known [33] that the entropy of small black hole, as attractor value of Eqn.(10), is given by, $S = 4\pi\sqrt{8\lambda_2}\sqrt{nw}$. This further matches with the statistical entropy $S = 4\pi\sqrt{nw}$ of corresponding microscopic configuration of counting the underlying string states, if $\lambda_2 = \frac{1}{8}$.

At the above attractor values of moduli fields, we find that the generalized attractor flow equations arising from Eqn.(10) are

$$\begin{aligned}
g_{11} &= \frac{1}{32} \frac{\Omega_{D-2}^2 \sqrt{16}(q_1^2 w^2 + n^2 q_2^2) (\frac{1}{4}\alpha' D^2 - \frac{5}{4}\alpha' D + \frac{3}{2}\alpha')^D}{\pi \alpha' (D-2)^2 (D-3)^2 G_N^2 n^2 w^2 \sqrt{nw}}, \\
g_{12} &= -\frac{1}{4} \Omega_{D-2} \frac{(-q_1^2 w^2 + n^2 q_2^2) (\frac{1}{4}\alpha' D^2 - \frac{5}{4}\alpha' D + \frac{3}{2}\alpha')^{D/2}}{(D-2)(D-3) G_N n^2 w^2} \sqrt{\frac{w}{n}}, \\
g_{22} &= \pi \frac{\alpha'}{n^2} \sqrt{\frac{2\lambda_2}{nw}} (3q_1^2 w^2 + n^2 q_2^2)
\end{aligned} \tag{14}$$

Therefore, we see that the TT component of the Hessian of the Sen's entropy function remains unchanged, under the change of dimensionality. At this point, it is noteworthy to mention from the geometric perspective of entropy function [?] that the small black hole thermodynamics, arising from the attractor fixed point configurations, show a degenerate system [34–37].

What follows further that the Hessian of Sen entropy function with respect to the moduli fields $\{S, T\}$ has standard attractor horizon properties. When it is evaluated at the an attractor fixed point of the small black hole configuration, one obtains the following factorization

$$\begin{pmatrix} g_{SS} & g_{ST} \\ g_{ST} & g_{TT} \end{pmatrix}_{attr} = K_{IJ}(S_{attr}, T_{attr}) S_{BH}(n, w); \tag{15}$$

where $\{K_{IJ}; I, J = S, T\}$ signify the second derivative of the Sen entropy function, and the subscripts S and T, as before, are understood as the respective partial derivatives with respect to the moduli fields u_S and u_T . This is the key result of this paper. Similar analysis arises from the standard attractor equations in the viewpoint of black hole effective potential energy. It has been mainly introduced in standard supergravity literature by Ferrara, Kallosh and Strominger [5–8, 20].

We anticipate that the higher derivative corrections do not contribute into the Hessian of the entropy function $F(u_S, u_T)$, and thus into the flow equations. This follows from the fact that the moduli $\{S, T\}$ appear only linearly in the higher derivative corrected small black hole entropy functions. Furthermore, we find that the higher order α' -corrections arising from the heterotic string compactification [31, 32] on $T^{9-D} \times S^1$ also do not enter in the Hessian matrix $F_{ij}(u_S, u_T)$ of entropy functions at any finite order.

Notice that the moduli aspects of solution which we show here is true for arbitrary D dimensional small black holes, *viz.*, the components $\{K_{IJ}; I, J = S, T\}$ determine the

moduli space properties of underlying heterotic string compactifications. In general, a D dimensional spacetime passing from the odd dimensions $D = 2m - 1$ to the even dimensions $D = 2m$ and then to the odd dimensions $D = 2m + 1$, it is known [33] that the horizon function f receives the following additional contribution

$$\Delta f = \frac{\Omega_{D-2}}{16\pi G_N} u_S v_1 v_2^{(D-2)/2} \alpha'^{m-2} \frac{(D-2)!}{(D-2m)!} v_2^{-m+1} \left(\lambda_{m-1} - \frac{2m\alpha'}{v_1} \lambda_m \right), \quad (16)$$

where one supposes for all $k = 1, \dots, m-1$ that λ_k may be determined from the lower-dimensional analysis. Thus, λ_m is the only free parameter to be determined at the chosen order of corrections. We notice further that the correction term Eqn.(16) is linear in u_S moduli. We find that such contributions do not manifestly enter into the Hessian of arbitrary higher derivative corrected entropy function.

Thus, the generalized flow equations as the equations of motion of Sen entropy function, and thereby factorization property of the Hessian are independent of dimensionality of the small black hole solutions. From the gravity side, we have shown that α' -corrections remain intact against the thermodynamic nature of moduli space attractor configurations. Such considerations may be envisaged to be of central importance in statistical physics [36, 37] and finite order higher derivative α' -contributions [38–40] into the black hole entropy. It is expected that there would exist a generic α' -corrected entropy formula, which would offer an intrinsic account of the statistical models arising from associated microscopic conformal field theories. At this point, it is worth mentioning that the interesting attractor inter-relations and their microscopic details are beyond the scope present set-up. Thus, we leave these issues for a future exploration.

Finally, it is worth mentioning that the higher fluxes are expected to play an important role in understanding the physics of black holes. In such cases, the moduli space geometry, Sen entropy function, flow equations and the factorization properties, e.g. Eqn.15, may be explored further at or way from the attractor fixed points. Further introspecting seems interesting towards an unified understanding of the entropy functions with an inclusion of non-zero higher fluxes.

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- [1] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” *Phys. Lett. B* **379** (1996) 99, [arXiv:hep-th/9601029](#).
 - [2] J. M. Maldacena, A. Strominger and E. Witten, “Black hole entropy in M-theory,” *JHEP* **9712** (1997) 002, [arXiv:hep-th/9711053](#).
 - [3] A. Sen, “Black Hole Solutions in Heterotic String Theory on a Torus,” *Nucl. Phys. B* **440** (1995) 421, [arXiv:hep-th/9411187v3](#).
 - [4] A. Sen, “Extremal Black Holes and Elementary String States,” *Mod. Phys. Lett. A* **10** (1995) 2081, [arXiv:hep-th/9504147v2](#).
 - [5] S. Ferrara, R. Kallosh and A. Strominger, “N=2 extremal black holes,” *Phys. Rev. D* **52** (1995) 5412, [arXiv:hep-th/9508072](#).
 - [6] A. Strominger, “Macroscopic Entropy of $N = 2$ Extremal Black Holes,” *Phys. Lett. B* **383** (1996) 39, [arXiv:hep-th/9602111](#).
 - [7] S. Ferrara and R. Kallosh, “Supersymmetry and Attractors,” *Phys. Rev. D* **54** (1996) 1514, [arXiv:hep-th/9602136](#).
 - [8] S. Ferrara and R. Kallosh, “Universality of Supersymmetric Attractors,” *Phys. Rev. D* **54** (1996) 1525, [arXiv:hep-th/9603090](#).
 - [9] G. W. Gibbons, R. Kallosh and B. Kol, “Moduli, scalar charges and the first law of black hole thermodynamics,” *Phys. Rev. Lett.* **77** (1996) 4992, [arXiv:hep-th/9607108](#).
 - [10] H. Ooguri, A. Strominger and C. Vafa, “Black hole attractors and the topological string,”

- Phys. Rev. D **70** (2004) 106007, [arXiv:hep-th/0405146](#).
- [11] B. Pioline, “Lectures on on black holes, topological strings and quantum attractors,” *Class. Quant. Grav.* **23** (2006) S981, [arXiv:hep-th/0607227](#).
 - [12] A. Sen, “Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity,” *JHEP* **0509** (2005) 038, [arXiv:hep-th/ 0506177](#).
 - [13] B. Sahoo, A. Sen, “BTZ Black Hole with Chern-Simons and Higher Derivative Terms,” ” *JHEP* **0607** (2008) 008, [arXiv:hep-th/0601228](#).
 - [14] B. Sahoo, A. Sen, “Higher Derivative Corrections to Non-supersymmetric Extremal Black Holes in N=2 Supergravity,” *JHEP* **0609** (2006) 029, [arXiv:hep-th/0603149](#).
 - [15] A. Sen, “Entropy Function for Heterotic Black Holes,” *JHEP* **0603** (2006) 008, [arXiv:hep-th/ 0508042](#).
 - [16] D. P. Jatkar, A. Sen, “Dyon Spectrum in CHL Models,” *JHEP* **0604** (2006) 018, [arXiv:hep-th/ 0510147](#).
 - [17] J. R. David, A. Sen, “CHL Dyons and Statistical Entropy Function from D1-D5 System,” *JHEP* **0611** (2006) 072, [arXiv:hep-th/ 0605210](#).
 - [18] A. Sen, “Black hole entropy function and the attractor mechanism in higher derivative gravity,” *JHEP* **0509** (2005) 038, [arXiv:hep-th/0506177](#).
 - [19] A. Dabholkar, A. Sen and S. Trivedi, “Black hole microstates and attractor without supersymmetry,” *JHEP* **0701** (2007) 096, [arXiv:hep-th/0611143](#).
 - [20] S. Ferrara, G. W. Gibbons and R. Kallosh, “Black holes and critical points in moduli space,” *Nucl. Phys. B* **500** (1997) 75, [arXiv:hep-th/9702103](#).
 - [21] S. Ferrara and R. Kallosh, “On $N = 8$ attractors,” *Phys. Rev. D* **73** (2006) 125005, [arXiv:hep-th/0603247](#).
 - [22] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, “Non-supersymmetric attractors,” *Phys. Rev. D* **72** (2005) 124021, [arXiv:hep-th/0507096](#).
 - [23] R. Kallosh, “New attractors,” *JHEP* **0512** (2005) 022, [arXiv:hep-th/0510024](#).
 - [24] P. K. Tripathy and S. P. Trivedi, “Non-supersymmetric attractors in string theory,” *JHEP* **0603** (2006) 022, [arXiv:hep-th/0511117](#).
 - [25] A. Giriyavets, “New attractors and area codes,” *JHEP* **0603** (2006) 020, [arXiv:hep-th/0511215](#).
 - [26] R. Kallosh, N. Sivanandam and M. Soroush, “The non-BPS black-hole attractor equation,”

- JHEP **0603** (2006) 060, [arXiv:hep-th/0602005](#).
- [27] R. D'Auria, S. Ferrara and M. Trigiante, "Critical points of the Black-Hole potential for homogeneous special geometries," JHEP **097** (2007) 0703, [arXiv:hep-th/0701090v1](#).
 - [28] S. Bellucci, S. Ferrara, M. Gunaydin and A. Marrani, "Charge Orbits of Symmetric Special Geometries and Attractors," Int. J. Mod. Phys. A **21** (2006) 5043, [arXiv:hep-th/0606209v2](#).
 - [29] A. Dabholkar, A. Sen and S. Trivedi, "Black Hole Microstates and Attractor Without Supersymmetry," JHEP **0701** (2007) 09, [arXiv:hep-th/0611143](#).
 - [30] A. Sen, "Stretching the Horizon of a Higher Dimensional Small Black Hole," JHEP **0507**, 073, (2005), [arXiv:hep-th/0505122](#).
 - [31] A. Dabholkar, "Exact counting of black hole microstates," Phys. Rev. Lett. **94** (2005) 241301, [arXiv:hep-th/0409148](#).
 - [32] A. Dabholkar, F. Denef, G. W. Moore and B. Pioline, "Precision counting of small black holes," JHEP **0510** (2005) 096, [arXiv:hep-th/0507014](#).
 - [33] P. Prester, "Lovelock type gravity and small black holes in heterotic string theory," JHEP **0602** (2006) 039, [arXiv:hep-th/0511306v1](#).
 - [34] B. N. Tiwari, "Sur les corrections de la géométrie thermodynamique des trous noirs", Quantum Gravity, Hoelback 2008, [arXiv:0801.4087v1 \[hep-th\]](#).
 - [35] T. Sarkar, G. Sengupta and B. N. Tiwari, "Thermodynamic Geometry and Extremal Black Holes in String Theory," JHEP **0810** (2008) 076, [arXiv:0806.3513v1 \[hep-th\]](#).
 - [36] S. Bellucci and B. N. Tiwari, "On the Microscopic Perspective of Black Branes Thermodynamic Geometry", Entropy **12** (2010) 2097-2143, [arXiv:0808.3921v1 \[hep-th\]](#).
 - [37] S. Bellucci and B. N. Tiwari, "State-space Correlations and Stabilities", Phys. Rev. D **82** (2010) 084008, [arXiv:0910.5309v1 \[hep-th\]](#).
 - [38] G. L. Cardoso, B. de Wit, J. Käppeli, T. Mohaupt, "Asymptotic degeneracy of dyonic N=4 string states and black hole entropy," JHEP **0412** (2004) 075, [arXiv:hep-th/0412287v1](#).
 - [39] G. L. Cardoso, "Black Holes, Entropy and String Theory", LMU-ASC 04/06, <http://www.asc.physik.uni-muenchen.de/research/preprints/index.html>, Proceedings RTN Meeting Corfu, September-2005.
 - [40] D. Astefanesei, K. Goldstein and S. Mahapatra, "Moduli and (un)attractor black hole thermodynamics," Gen. Rel. Grav. **40** (2008) 2069, [arXiv:hep-th/0611140](#).